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Relativistic Orbits of Classical Charged Bodies in a Spherically Symmetric Electrostatic Field

Hartmut Frommert[1]

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The orbits of a relativistic charged body in a static, spherically symmetric electric field are calculated and classified in the classical theory. Contrary to the nonrelativistic problem, we find that there is a limiting minimal value for the angular momentum L_c. Should the actual angular momentum of a charged test body be lower than this limit, the test particle will spiral into the central point charge instead of having (precessing) Keplerian orbits.

1. INTRODUCTION

Within Dirac's theory or Sommerfeld's semiclassical theory for the fine structure of the hydrogen atom it is well known that the ground state will become unstable for nuclear charges (of hydrogen-like ions) $Z > 1/\alpha$ (see, e.g., Greiner, 1981). It is the aim of this paper to show that this phenomenon is not a typical quantum mechanical one, but that also in the classical theory of a relativistic charged point particle within a static Coulomb field there exist nonstable orbits connected with the existence of a critical angular momentum L_c; should the actual angular momentum of the point particle be lower than L_c, the particle will spiral into the central point charge independent of its energy (unstable orbit), while the nonrelativistic treatment yields Keplerian orbits in any case (ellipses, parabolas, and hyperbolas). As in the quantum mechanical case mentioned above, we neglect the radiative reaction force on the charged point particle.

2. LAGRANGE FUNCTION AND THE EQUATION OF MOTION

Neglecting the radiative backreaction, the Lagrange function of a relativistic point particle (rest mass m_0, charge e) in an underlying electromagnetic potential A_μ is given by

[1] Department of Physics, University of Constance, D-78434 Konstanz, Germany; e-mail: Hartmut.Frommert@uni-konstanz.de.

$$\mathcal{L} = -m_0c^2\sqrt{\eta_{\mu\nu}v^\mu v^\nu} - \frac{e}{c}\,v^\mu A_\mu \tag{1}$$

v^μ is the timelike 4-velocity, and $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. In the following, the magnetic potential A_i, $i = 1, \ldots, 3$, is assumed to vanish, and $A_0 = c\Phi$, where Φ is the electrostatic potential and a function of the radial coordinate r only. Then the Lagrange function (1) simplifies to

$$\mathcal{L} = -m_0c^2\sqrt{1 - \frac{v^2}{c^2}} - e\Phi(r) \tag{2}$$

where v is the absolute value of the 3-velocity. Because of the spherical symmetry of Φ there exists angular momentum conservation, and consequently the motion of the particle will take place in a plane orthogonal with respect to the angular momentum vector; we *choose* as this plane the x–y plane.

Using plane polar coordinates r and φ, we find that v^2 simplifies to

$$v^2 = \dot{r}^2 + r^2\dot{\varphi}^2 \tag{3}$$

and we can derive the Euler–Lagrange equations for $r(t)$ and $\varphi(t)$. In the case of φ we have, since $\partial\mathcal{L}/\tau\varphi \equiv 0$, a conserved angular momentum L, i.e.,

$$L = \frac{m_0 r^2\dot{\varphi}}{\sqrt{1 - v^2/c^2}} = \mathrm{const} \tag{4}$$

Inserting v^2 from (3) and resolving for $\dot{\varphi}$, we get

$$\dot{\varphi} = \frac{c}{r}\sqrt{\frac{1 - \dot{r}^2/c^2}{1 + (m_0cr/L)^2}} \tag{5}$$

Since $\mathcal{L}$ is not explicitly dependent on t, we have, in addition, energy conservation, i.e., Hamilton's function is a constant; this results in

$$E = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} + e\Phi = \mathrm{const} \tag{6}$$

Inserting (3) and resolving for $\dot{r}$ yields

$$\dot{r} = c\sqrt{1 - \left(\frac{m_0c^2}{E - e\Phi}\right)^2 - \left(\frac{r\dot{\varphi}}{c}\right)^2} \tag{7}$$

Eliminating $\dot{\varphi}$ by equation (5), one obtains after a short calculation

$$\dot{r} = c\sqrt{1 - \left(\frac{m_0c^2}{E - e\Phi}\right)^2\left[1 + \left(\frac{L}{m_0cr}\right)^2\right]} \tag{8}$$

Herewith equation (5) takes the form

$$\dot{\varphi} = \frac{Lc^2}{(E - e\Phi)r^2} \tag{9}$$

By combination of (7) and (9) we get the differential equation for calculating the orbit, i.e., $r(\varphi)$:

$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = r^2 \frac{m_0 c}{L} \sqrt{\left(\frac{E - e\Phi}{m_0 c^2}\right)^2 - 1 - \left(\frac{L}{m_0 c r}\right)^2} \tag{10}$$

Substituting $r = 1/s$, we can rewrite this as

$$\frac{ds}{d\varphi} = -\frac{m_0 c}{L} \sqrt{\left(\frac{E - e\Phi}{m_0 c^2}\right)^2 - 1 - \left(\frac{L}{m_0 c}\right)^2 s^2} \tag{11}$$

For the Coulomb potential $\Phi = Q/r = Qs$ we get finally

$$\frac{ds}{d\varphi} = -\frac{m_0 c}{L} \left\{\left[\left(\frac{E}{m_0 c^2}\right)^2 - 1\right] - 2 \frac{E}{m_0 c^2} \frac{eQ}{m_0 c^2} s - \left[\left(\frac{L}{m_0 c}\right)^2 - \left(\frac{eQ}{m_0 c^2}\right)^2\right] s^2\right\}^{1/2} \tag{12}$$

Looking now for a more suggestive form of the orbital differential equation, one can take the square of this equation, differentiate with respect to φ, and divide by $2\, d\varphi/ds$. Thus one obtains

$$\frac{d^2 s}{d\varphi^2} = -\left[1 - \left(\frac{eQ}{cL}\right)^2\right](s \pm s_0) \tag{13}$$

with

$$s_0 = \frac{(E/m_0 c^2)(|eQ|/m_0 c^2)(m_0 c/L)^2}{1 - (eQ/cL)^2} \tag{14}$$

The upper sign in (13) applies if e and Q have the same sign, and the lower sign applies otherwise. Depending on the value of L, equation (13) has three different types of solutions:

1. For $L > L_c$: periodic solutions (trigonometric functions) in φ.
2. For $L = L_c$: a limiting algebraic case.
3. For $L < L_c$: nonperiodic (hyperbolic and exponential functions).

Where the "critical" angular momentum L_c is given by

$$L_c = \left|\frac{eQ}{c}\right| \tag{15}$$

For easier processing, it is appropriate to substitute three new constants for E, L, and Q (or L_c):

$$u := \frac{E}{m_0 c^2}, \qquad l := \frac{L}{m_0 c}, \qquad l_c := \frac{L_c}{m_0 c} = \frac{|eQ|}{m_0 c^2} \tag{16}$$

where u is a dimensionless "specific energy," while the "specific angular momentum" l and the "specific critical angular momentum" l_c have the dimension of a length. The length l_c gives the distance at which the absolute value of the electric potential energy eQ/r of the test particle (i.e., between the charges e and Q) is equal to the rest energy $m_0 c^2$ of the particle. This demonstrates the special-relativistic nature of the effects where l_c or L_c plays a role. Then equation (12) reads

$$\frac{ds}{d\varphi} = -\frac{1}{l}\sqrt{(u^2 - 1) \mp 2ul_c s - (l^2 - l_c^2)s^2} \tag{17}$$

The constant s_0 in (14) takes the value

$$s_0 = \frac{ul_c}{l^2 - l_c^2} \tag{18}$$

3. INTEGRATION OF THE EQUATION OF MOTION

3.1. Large Angular Momentum $L > L_c$: Keplerian Orbits

In the case $l > l_c$ or $L > L_c = |eQ|/c$ we can rewrite equation (17) in the following form using (18):

$$\frac{ds}{d\varphi} = -\frac{1}{l}\sqrt{(u^2 - 1) - (l^2 - l_c^2)(s^2 \pm 2s_0 s)} \tag{19}$$

After quadratic completion we obtain

$$\begin{aligned}\frac{ds}{d\varphi} &= -\frac{1}{l}\{[(u^2 - 1) + (l^2 - l_c^2)s_0^2] - (l^2 - l_c^2)(s \pm s_0)^2\}^{1/2} \\ &= -\frac{1}{l}\left\{\left(\frac{u^2 l^2}{l^2 - l_c^2} - 1\right)\cdot\left[1 - \frac{(l^2 - l_c^2)^2(s \pm s_0)^2}{(u^2 - 1)l^2 + l_c^2}\right]\right\}^{1/2}\end{aligned} \tag{20}$$

With respect to the radicand we can substitute

$$s = \mp s_0 + \frac{\sqrt{(u^2 - 1)l^2 + l_c^2}}{l^2 - l_c^2}\cos\alpha \tag{21}$$

which yields from (20)

$$\frac{d\alpha}{d\varphi} = \sqrt{1 - \left(\frac{l_c}{l}\right)^2} \Rightarrow \alpha = \sqrt{1 - \left(\frac{l_c}{l}\right)^2}(\varphi - \varphi_0) \tag{22}$$

Combining (21) and (22), we obtain for $r(\varphi)$:

$$r(\varphi) = \frac{r_0}{\mp 1 + \epsilon \cos[\sqrt{1 - (l_c/l)^2}(\varphi - \varphi_0)]} \tag{23}$$

where

$$\epsilon = \frac{\sqrt{(u^2 - 1)l^2 + l_c^2}}{ul_c} = \sqrt{\frac{1}{u^2} + \left(\frac{l}{l_c}\right)^2 - \left(\frac{l}{ul_c}\right)^2} \tag{24}$$

is the numerical eccentricity. The upper sign applies in the repulsive and the lower one in the attractive case.

The solution (23) represents a *precessing* Keplerian orbit, i.e., a precessing ellipse, parabola, or hyperbola, depending on the value of $\epsilon < 1, = 1, > 1$, respectively, where in case of a repulsive force $\epsilon > 1$ must hold because of $r \geq 0$. In the case of bound states it is a periodic orbit with period $2\pi/[1 - (l_c/l)^2]^{1/2} > 2\pi$; thus we have a progressive periapsis shift of

$$\delta\varphi = 2\pi\left(\frac{1}{\sqrt{1 - (l_c/l)^2}} - 1\right) \tag{25}$$

per cycle caused by the critical value of l_c. For large values of L or l, this expression can be approximated by

$$\delta\varphi \approx \pi\left(\frac{l_c}{l}\right)^2 = \pi\left(\frac{|eQ|}{Lc}\right)^2 \tag{26}$$

In case of scattering states $\epsilon > 1$, we have a precessing hyperbola, which means that the asymptotes calculated from (23) are given by

$$\varphi - \varphi_0 = \pm\frac{1}{\sqrt{1 - (l_c/l)^2}} \arccos\left(\mp\frac{1}{\epsilon}\right) \tag{27}$$

This precessing hyperbola does not coincide with an exact hyperbola with another eccentricity. Instead, its asymptotes, together with its apsid line, precess progressively while the particle moves around the center, at exactly the same rate per passed angle as for the ellipses in the bound case above, so that from approach to escape the trajectory has precessed by a total of

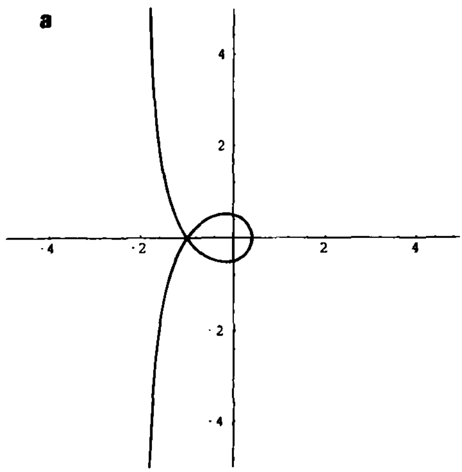
$$\delta\varphi = \left(\frac{1}{\sqrt{1 - (l_c/l)^2}} - 1\right)\cdot 2 \arccos\left(\mp\frac{1}{\epsilon}\right) \tag{28}$$

If the specific angular momentum l comes close to the limit l_c, the angle $\delta\varphi$

will become larger and larger, so that the particle may orbit the central charge one or more times before it escapes again to infinity (see Fig. 1).

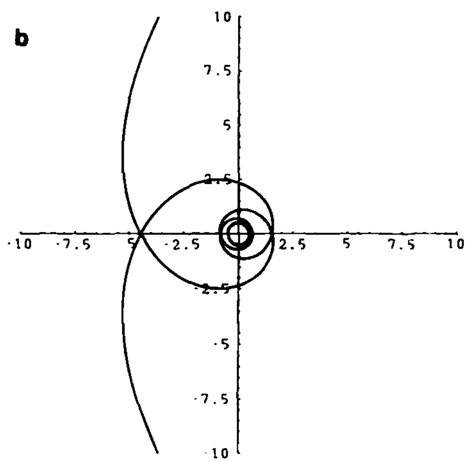
Such types of orbits are also known from general relativity, where both massive and massless test bodies with small angular momentum have similar trajectories within Schwarzschild's metric gravitational field (see, e.g., von Laue, 1965, §45, and Misner *et al.*, 1973, Chapter 25.6). However, as stated above, our results are a purely special-relativistic effect, while in the Schwarz-

a



$$\epsilon = \sqrt{2} = 1.41, \; L = 1.15 L_c.$$

b



$$\epsilon = 1.1, \; L = 1.0114 L_c.$$

Fig. 1. The test particle may orbit the central charge one (a) or more times (b) before it escapes again to infinity if the angular momentum L approaches L_c.

schild case the nonlinear structure of the gravitational field has an additional impact.

3.2. The Limiting Case $L = L_c$: Quadratic Spiral Trajectories

For $L = L_c$, i.e., $l = l_c$ (corresponding to $s_0 = \infty$), equation (17) takes the form

$$\frac{ds}{d\varphi} = \sqrt{\frac{2u}{l_c}(s_1 \mp s)} \tag{29}$$

with

$$s_1 = \frac{u^2 - 1}{2ul_c} \tag{30}$$

(the upper sign is valid for repulsive forces, the lower one for the attractive case). The solution reads

$$s = \frac{u}{2l_c}(\varphi - \varphi_0)^2 \pm s_1 \tag{31}$$

According to (30), s_1 is positive, 0, or negative as $u > 1$, $= 1$, or < 1, respectively. Then the trajectory is given by

$$r = \mp \frac{r_0}{a(\varphi - \varphi_0)^2 \mp 1} \tag{32}$$

where

$$r_0 = \frac{2ul_c}{|u^2 - 1|}, \qquad a = \frac{u^2}{|u^2 - 1|}$$

Since r must be positive, it is valid for:

- $u < 1$ and attractive forces: The trajectory has a maximal distance r_0 from the origin at $\varphi = \varphi_0$ and spirals quadratically into the origin as $|\varphi - \varphi_0| \to \infty$.
- $u > 1$ and attractive forces:

$$|\varphi - \varphi_0| > \frac{1}{\sqrt{a}} = \frac{\sqrt{u^2 - 1}}{u} \tag{33}$$

  The trajectory comes from infinity at $|\varphi - \varphi_0| = 1/\sqrt{a}$ and spirals quadratically into the origin for $|\varphi - \varphi_0| \to \infty$.
- $u > 1$ and repulsive forces:

$$|\varphi - \varphi_0| < \frac{1}{\sqrt{a}} = \frac{\sqrt{u^2 - 1}}{u} \tag{34}$$

In this case, $r_0 = r(\varphi = \varphi_0)$ represents the minimal distance from the central charge and for $|\varphi - \varphi_0| \to 1/\sqrt{a}$ the trajectory runs out to infinity without any spiraling.

No solution exists in the repulsive case for $u < 1$. For $u = 1$ ($s_1 = 0$), we have only a solution in the attractive case, namely

$$s = (u/2l_c)(\varphi - \varphi_0)^2 \tag{35}$$

which comes from infinity at $\varphi = \varphi_0$ and spirals quadratically into the origin with $|\varphi - \varphi_0| \to \infty$.

3.3. Small Angular Momentum $L < L_c$: Exponential Spiral Orbits

In the last case $L < L_c$, i.e., $l < l_c$, we can rewrite equation (17) as

$$\left(\frac{ds}{d\varphi}\right)^2 l^2 = \left(1 + \frac{u^2 l^2}{l_c^2 - l^2}\right)\left[\frac{(l_c^2 - l^2)^2}{(u^2 - 1)l^2 + l_c^2}(s \mp s_0)^2 - 1\right] \tag{36}$$

[s_0 according to (18) and upper sign for repulsive, the lower one for attractive forces]. In the following, it is convenient to introduce the abbreviation

$$b = \sqrt{(l_c/l)^2 - 1} \tag{37}$$

which is a positive real constant. We discuss the solutions of this equation of motion for the *attractive case* ($eQ < 0$) first. Then we have to distinguish three cases, corresponding to the value of the "specific energy" constant u:

A. $u < 1$, i.e., sum of kinetic and potential energy negative: In this case the solution of (36) reads (see Fig. 2)

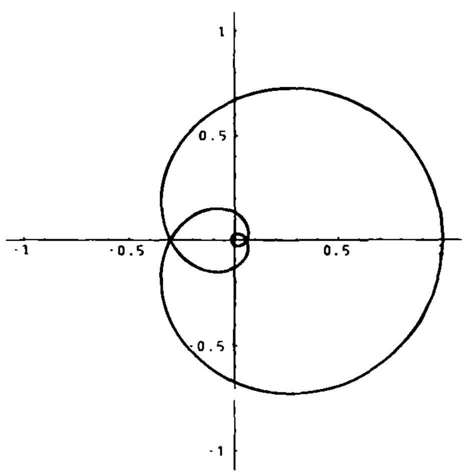


Fig. 2. Orbits for $L < L_c$, $u < 1$ (bound states).

$$r = \frac{r_0}{1 + a\{\cosh[b(\varphi - \varphi_0)] - 1\}} \tag{38}$$

where

$$r_0 = \frac{l_c^2 - l^2}{\sqrt{l_c^2 - (1 - u^2)l^2} - ul_c} > 0 \tag{39}$$

$$a = \frac{1}{1 - ul_c/\sqrt{l_c^2 - (1 - u^2)l^2}} > 1 \tag{40}$$

The trajectory described by equation (38) has its greatest distance r_0 from the origin at $\varphi = \varphi_0$ and spirals exponentially into the central charge for both $\varphi \to \infty$ and $\varphi \to -\infty$ as

$$r \to \frac{2r_0}{a} e^{-b|\varphi - \varphi_0|} \tag{41}$$

B. $u = 1$, i.e., sum of kinetic and potential energy zero: The solution of (36) takes the form (Fig. 3)

$$r = \frac{r_1}{\cosh[b(\varphi - \varphi_0)] - 1} \tag{42}$$

with

$$r_1 = \frac{l_c^2 - l^2}{l_c} > 0 \tag{43}$$

The trajectory approaches infinity for $\varphi = \varphi_0$ and spirals exponentially into the origin for $\varphi \to \infty$ as

$$r \to 2r_1 e^{-b|\varphi - \varphi_0|} \tag{44}$$

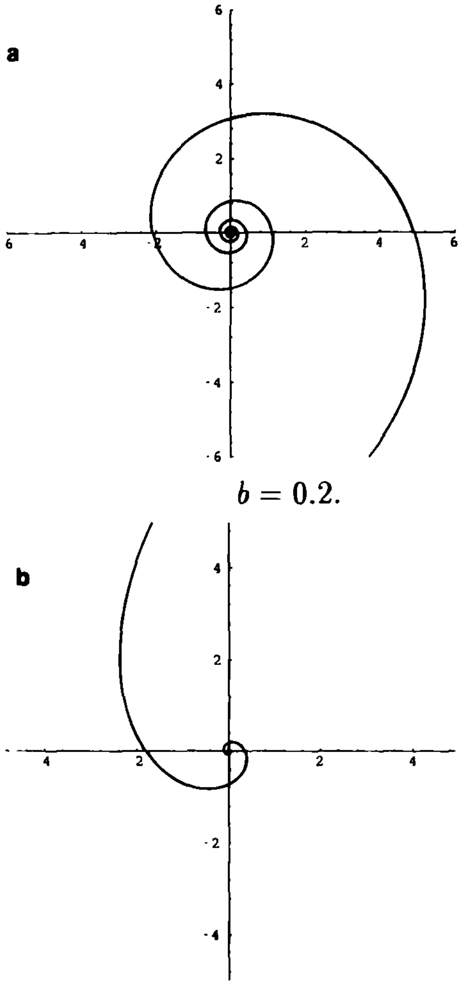
C. $u > 1$, i.e., sum of kinetic and potential energy positive: Equation (36) has the solution (Fig. 4)

$$r = \frac{r_0}{a\{\cosh[b(\varphi - \varphi_0)] - 1\} - 1} \tag{45}$$

with

$$r_0 = \frac{l_c^2 - l^2}{ul_c - \sqrt{l_c^2 - (1 - u^2)l^2}} > 0 \tag{46}$$

$$a = \frac{1}{ul_c/\sqrt{l_c^2 - (1 - u^2)l^2} - 1} > 0 \tag{47}$$



$b = 0.32.$

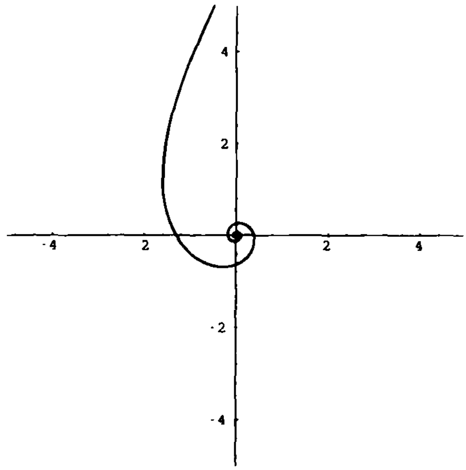
Fig. 3. Orbits for $L < L_c$, $u = 1$ (limiting case).

Because the distance r must be always positive, the range of φ is restricted so that the denominator in (45) remains positive:

$$|\varphi - \varphi_0| > \frac{1}{b} \operatorname{arcosh}\left(1 + \frac{1}{a}\right) = \varphi_\infty \tag{48}$$

The trajectory does not enter the φ interval $\varphi_0 - \varphi_\infty < \varphi < \varphi_0 + \varphi_\infty$. It comes from infinity at

$$\varphi = \varphi_0 \pm \varphi_\infty$$



$a = 1.5,\ b = 0.15.$

Fig. 4. Orbits for $L < L_c$, $u > 1$ (scattering states).

and spirals exponentially into the origin for $|\varphi - \varphi_0| \to \infty$ as

$$r \to \frac{2r_0}{a}\, e^{-b|\varphi-\varphi_0|} \tag{49}$$

i.e., exactly as in the first case.

In a second step we discuss the motion for the repulsive case. For this, the solution of (36) reads

$$r = \frac{r_0}{1 - a\{\cosh[b(\varphi - \varphi_0)] - 1\}} \tag{50}$$

with

$$r_0 = \frac{l_c^2 - l^2}{ul_c - \sqrt{l_c^2 - (1 - u^2)l^2}} \tag{51}$$

$$a = \frac{1}{ul_c/\sqrt{l_c^2 - (1 - u^2)l^2} - 1} \tag{52}$$

Because of $r > 0$ it follows that $u > 1$ must hold. The trajectory has a *minimal* distance $r_0 = r(\varphi_0)$ and goes to infinity without any spiraling for

$$|\varphi - \varphi_0| \to \varphi_\infty = \frac{1}{b}\,\operatorname{arcosh}\left(\frac{ul_c}{\sqrt{l_c^2 - (1 - u^2)l^2}}\right)$$

Table I. Attractive Case

	$E < m_0c^2$ (bound states)	$E = m_0c^2$	$E > m_0c^2$ (scattering states)
$L > L_c$	Precessing ellipse, Eq. (23), $0 \leq \epsilon < 1$	Precessing parabola, Eq. (23), $\epsilon = 1$	Precessing hyperbola, Eq. (23), $\epsilon > 1$
$L = L_c$	Quadratical spiral from maximal r_0 into the center, Eq. (32), lower signs	Quadratic spiral from infinity into the center, Eq. (35)	Quadratic spiral from infinity into the center, Eq. (32), plus sign before RHS, minus sign in denominator
$L < L_c$	Exponential spiral from maximal r_0 into the center, Eq. (38), Fig. 2	Exponential spiral from infinity into the center, Eq. (42), Fig. 3	Exponential spiral from infinity into the center, Eq. (45), Fig. 4

It never leaves the φ interval $\varphi_0 - \varphi_\infty < \varphi < \varphi_0 + \varphi_\infty$. It comes from infinity at $\varphi = \varphi_0 - \varphi_\infty$, has its closest approach, $r = r_0$, at $\varphi = \varphi_0$, and leaves again to infinity at $\varphi = \varphi_0 - \varphi_\infty$.

4. SUMMARY

The different classes of orbits discussed above for the special-relativistic Coulomb problem are summarized in Tables I and II.

Obviously, in classical special-relativistic electrostatics, there exists a limiting angular momentum: If a charged body which is attracted by the electric field has less angular momentum than this critical value, it will spiral into the source, and this already without taking into account the radiative energy losses. The physical meaning is that low-angular-momentum test charges get so close to the central charge, i.e., into such a strong field, that they are accelerated to relativistic velocities and can then no longer escape.

Besides the fact that this result is of interest on its own, one may look for applications, which can be expected in that part of physics where special

Table II. Repulsive Case

	$E > m_0c^2$ (scattering states)
$L > L_c$	Precessing repulsive hyperbola, Eq. (23), upper sign, $\epsilon > 1$
$L = L_c$	Quadratic approach to and escape from minimal r_0 to infinity, Eq. (32), upper signs
$L < L_c$	Exponential approach to and escape from minimal r_0 to infinity, Eq. (50)

relativity plays a role, while quantum effects stay weak. However, with respect to applications, the radiative backreaction will be important and must be taken into account. This will be done in a subsequent paper. Nevertheless we will give an estimation of the critical situation discussed above: For electrons as test particles, it is necessary to concentrate a charge of

$$Q = l_c m_0 c^2/e \approx 4.2 \times 10^{12} e(l_c/\text{cm})$$

in a volume of a radius smaller than l_c, i.e., for a 1-cm radius, more than 4×10^{12} elementary charges have to be stabilized and localized in this volume, generating a voltage at its surface which corresponds to the electron's m_0c^2/e, i.e., more than 5.11×10^5 V; "classical" test bodies would require even a much higher central charge. It may be difficult to realize such a densely packed charge. However, it is the hope that the radiative reaction force will improve the experimental conditions.

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